

From spin to anyon notation: The XXZ Heisenberg model as a D_3 (or $su(2)_4$) anyon chain

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Abstract

We discuss a relationship between certain one-dimensional quantum spin chains and anyon chains. In particular we show how the XXZ Heisenberg chain is realised as a D_3 (alternately $su(2)_4$) anyon model. We find the difference between the models lie primarily in choice of boundary condition.

1 Introduction

The purpose of this letter is to note the correspondence between certain one-dimensional spin and anyon chains. We consider spin chains that have the underlying symmetry of a quasi-triangular Hopf algebra (e.g. a quantum group) [7, 17]. Consequently, it is natural to discuss integrable models constructable from the Quantum Inverse Scattering Method (QISM) [15, 21] and its variants.

On the other hand, there are anyon chains. These were constructed as analogues of spin chains [10]. One difference is that anyons are not required to have integer quantum dimension, which is, roughly speaking, the dimension of the internal Hilbert space of the particle and determines the probability that fusion leads to annihilation or creation of other anyons [19]. The Hilbert spaces used for the anyon chains often have no tensor product structure. These chains have also been shown to exhibit topological symmetries [4, 10]. It is possible to construct anyonic models from a quasi-triangular Hopf algebra [9].

An equivalence between spin and anyon chains occurs when the underlying symmetry of each is that of the same quasi-triangular Hopf algebra. To illustrate this correspondence we present the nearest-neighbour XXZ Heisenberg chain viewed as a D_3 anyon model. While the local Hamiltonian has the complete D_3 symmetry, the symmetry of the global Hamiltonian depends upon the boundary conditions imposed. Thus the correspondence depends upon the boundary conditions. We consider open boundaries with free ends [20], periodic boundaries of both spin [8, 25] and anyon type [10, 23], and braided boundaries [12, 14, 16]. Of these only the open and braided boundary conditions always have an equivalent description in the spin and anyon pictures.

It is also possible to present the XXZ model using other underlying symmetries, e.g. $su(2)_4$, D_5 or $U_q(su(2))$, however, using D_3 has certain advantages. Firstly, there are no superfluous anyons, like the anyons in half-integer subsector of $su(2)_4$ or an additional anyon of quantum dimension two in D_5 . Secondly, the anisotropy parameter is not dependent upon the algebra

like $U_q(su(2))$ where $\frac{J_z}{J} = \cosh(\ln(q))$ (J and J_z are the coupling constants of the model). We also note that the XXZ Heisenberg chain has appeared in other papers in anyonic form, specifically as the spin-1 $su(2)_4$ chain [11, 24], although not discussed as such.

2 Background

Here we present the background information for the XXZ Heisenberg model, the D_3 algebra and the spin and anyon bases for the models. We also discuss when the operators in each of the bases are said to have the symmetry of D_3 .

The algebra

The group D_3 is the symmetries on a triangle consisting of a rotation, σ , and flip, τ . The group has the presentation,

$$D_3 = \{\sigma, \tau | \sigma^3 = \tau^2 = \sigma\tau\sigma\tau = 1\}.$$

Its group algebra is the linear combination of its elements over the complex numbers. It is also possible to embed this algebra into a k -fold space by use of the general coproduct,

$$\Delta^{(k)}(g) = \overbrace{g \otimes \dots \otimes g}^{k\text{-times}}, \quad g \in D_3,$$

extended linearly to the algebra. This algebra is known to form a quasi-triangular Hopf algebra [7, 17]. As it is cocommutative the universal R -matrix is just the identity. The representation theory of this algebra is also known, it has three irreducible representations, two are one-dimensional,

$$\pi_{\pm}(\sigma) = 1, \quad \pi_{\pm}(\tau) = \pm 1,$$

and one is two dimensional,

$$\pi_2(\sigma) = \begin{pmatrix} e^{\frac{2i\pi}{3}} & 0 \\ 0 & e^{-\frac{2i\pi}{3}} \end{pmatrix}, \quad \pi_2(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For each representation π_a we will associated a space (module) V_a . The fusion rules are as follows:

$$V_- \otimes V_- \cong V_+, \quad V_- \otimes V_2 \cong V_2, \quad \text{and} \quad V_2 \otimes V_2 \cong V_2 \oplus V_+ \oplus V_-.$$

The space V_+ is the vacuum or trivial space and fusion with it is trivial, i.e. $V_+ \otimes V_a \cong V_a$. As fusion is associative i.e. $(V_a \otimes V_b) \otimes V_c \cong V_a \otimes (V_b \otimes V_c)$, there exists F -moves which relate the two different ways to decompose the 3-fold tensor product space [18, 23]. The interpretation of these operators can be understood via the diagrammatic interpretation presented later. The F -moves can be explicitly constructed from the representations above and are found in the appendix of this letter.

To construct an anyonic model we associate with each D_3 module an anyon [9]. The fusion of anyons is governed by the fusion of the respective modules. Similarly the other properties of the anyons are inherited from the representation theory of the algebra.

The Bases

The Hilbert space we shall use consists of \mathcal{L} 2-state (spin- $\frac{1}{2}$) sites and an auxiliary \mathbb{C}^4 space, specifically

$$[V_2 \oplus V_+ \oplus V_-] \otimes V_2^{\otimes \mathcal{L}}.$$

This has a natural basis with $2^{\mathcal{L}+2}$ vectors, which we shall refer to as the spin basis. We also note that if we project onto V_+ component of the auxiliary space then we are just left with \mathcal{L} spin- $\frac{1}{2}$ sites.

To write the anyon basis we need to consider fusion paths [18, 19]. Working from left to right we fuse an irreducible subspace in the auxiliary space with V_2 , choosing which irreducible space we project onto, this is then fused to another V_2 , so on and so forth. We record the irreducible subspace of the auxiliary space and the subsequent irreducible subspaces which appear after fusion,

$$\begin{aligned}
& a_0 \text{ --- } \overset{2}{\underset{|}{\text{---}}} \underset{a_1}{\text{---}} \overset{2}{\underset{|}{\text{---}}} \underset{a_2}{\text{---}} \overset{2}{\underset{|}{\text{---}}} \underset{a_3}{\text{---}} \text{ --- } \text{ --- } \overset{2}{\underset{|}{\text{---}}} \underset{a_{\mathcal{L}-1}}{\text{---}} \overset{2}{\underset{|}{\text{---}}} a_{\mathcal{L}} \\
& \equiv ((\cdot((V_{a_0} \otimes V_2)_{a_1} \otimes V_2)_{a_2} \cdot \cdot)_{a_{\mathcal{L}-1}} \otimes V_2)_{a_{\mathcal{L}}} \\
& \equiv |a_0 a_1 a_2 \dots a_{\mathcal{L}-1} a_{\mathcal{L}}\rangle.
\end{aligned}$$

Here the use of the subscript of the bracket, $(\cdot)_a$, denotes the subspace of the space inside the bracket isomorphic V_a . These sequences of labels form the basis vectors of the anyon Hilbert space. Using this formulation we observe these anyon basis vectors correspond to a subspace in the spin basis, whose dimensionality is equal to that of dimension of the out-going anyon¹. We remark that each label a_i , as it is produced by fusion, is limited by the preceding label. We find that neighbouring pairs must belong to the following set,

$$a_i a_{i+1} \in \{+2, -2, 2+, 2-, 22\}.$$

We now have an appropriate anyon basis.

Diagrammatically we have fused left to right, however we can rearrange fusion adopting the additional convention of also fusing top to bottom. The reordering of fusion is done by the aforementioned F -moves. In terms of fusion diagrams we have,

$$a \text{ --- } \overset{b}{\underset{|}{\text{---}}} \underset{d}{\text{---}} \overset{c}{\underset{|}{\text{---}}} e = \sum_{d'} (F_e^{abc})_{d'}^d a \text{ --- } \overset{b}{\underset{\diagup}{\text{---}}} \underset{d'}{\text{---}} \overset{c}{\underset{\diagdown}{\text{---}}} e$$

On the left the anyons a and b are fused with the result fusing to the the anyon c , while on the right the anyons b and c are fused with the result fusing to a . The F -moves must satisfy a pentagon equation, although in the D_3 case this is automatically satisfied as D_3 forms a Hopf algebra.

An operator can be expressed in both the spin and anyon bases provided that it has the underlying symmetry of the D_3 algebra. Suppose we have an operator, \mathcal{O} , in the spin basis. It is said to have D_3 symmetry if it commutes with the action of the algebra,

$$[\Pi(g), \mathcal{O}] = 0 \quad \text{where} \quad \Pi = \left([\pi_2 \oplus \pi_+ \oplus \pi_-] \otimes \pi_2^{\otimes \mathcal{L}} \right) \circ \Delta^{(\mathcal{L}+1)}, \quad (1)$$

for all $g \in D_3$. This operator will have a counterpart in the anyon basis which we will denote $\tilde{\mathcal{O}}$. The D_3 symmetry is both sufficient and necessary. As the operator commutes with the action of the algebra Schur's lemma requires:

$$\text{If } a_{\mathcal{L}} \neq a'_{\mathcal{L}} \quad \text{then} \quad \langle a'_0 a'_1 \dots a'_{\mathcal{L}} | \tilde{\mathcal{O}} | a_0 a_1 \dots a_{\mathcal{L}} \rangle = 0. \quad (2)$$

¹While an individual anyon basis vector will correspond to a subspace in the spin basis, generic vectors in the anyon Hilbert space will have no such correspondence.

Thus for an operator, $\tilde{\mathcal{O}}$, in the anyon basis to have a spin counterpart the last label, $a_{\mathcal{L}}$, must be invariant under the action of $\tilde{\mathcal{O}}$. Similarly, from construction the auxiliary space must be also invariant under the action of operator and thus the first label a_0 is invariant under $\tilde{\mathcal{O}}$. The fixing of a_0 and $a_{\mathcal{L}}$ are necessary and sufficient for an operator in the anyon basis to have a counterpart in the spin basis². This is the same as the operator having hidden quantum group symmetry [22]. Any such operator which acts non-trivially on k neighbouring sites (in the spin basis) will have an anyon counterpart that acts on $k + 1$ labels, e.g. for $h \in V_2 \otimes V_2$ we have the equivalence $h_{i(i+1)} \leftrightarrow \tilde{h}_{(i-1)i(i+1)}$.

We also want to determine the dimensionality of the anyon Hilbert space. We define the number $N_{\mathcal{L}}^{(ab)}$ to be the number of basis vectors with $a_0 = a$ and $a_{\mathcal{L}} = b$. The numbers are determined by the relations

$$N_{\mathcal{L}}^{(a+)} = N_{\mathcal{L}}^{(a-)} = N_{\mathcal{L}-1}^{(a2)}, \quad N_{\mathcal{L}-1}^{(ab)} = N_{\mathcal{L}-1}^{(ba)} \quad \text{and} \quad N_{\mathcal{L}}^{(a2)} = N_{\mathcal{L}-1}^{(a2)} + 2N_{\mathcal{L}-1}^{(a+)}.$$

We can recognise that these numbers are those appearing in the Jacobsthal sequence (A001045 [1]) which have the form

$$N_{\mathcal{L}}^{(22)} = \frac{1}{3} (2^{\mathcal{L}+1} + (-1)^{\mathcal{L}}).$$

Using this we can determine the dimension of the anyon Hilbert space,

$$\text{Anyon dimension} = \sum_{a,b} N_{\mathcal{L}}^{(ab)} = N_{\mathcal{L}+2}^{(22)} = \frac{1}{3} (2^{\mathcal{L}+3} + (-1)^{\mathcal{L}}).$$

We can also determine that we indeed have the correct dimension of the spin Hilbert space

$$\text{Spin dimension} = \sum_{a,b} N_{\mathcal{L}}^{(ab)} \times \dim(V_b) = 2^{\mathcal{L}+2}.$$

Projection Operators and the Local Hamiltonian

As we are dealing with a model with D_3 symmetry we expect that the global Hamiltonian will just be composed of projection operators. As we only consider models with nearest-neighbour interactions we further restrict ourselves to projection operators on two sites. In the spin basis we have the 2-site projection operator is given by,

$$P^{(b)} = \frac{\dim(V_b)}{6} \sum_{g \in D_3} \text{Trace}(\pi_b(g^{-1})) \pi_2(g) \otimes \pi_2(g).$$

By construction this local operator commutes with the action of the algebra and has a corresponding operator in the anyon basis. We can diagrammatically determine the projection operator in the following way [10],

$$\begin{aligned} \tilde{P}_{i-1,i,i+1}^{(b)} \left\{ \begin{array}{c} 2 \quad 2 \\ | \quad | \\ \hline a_{i-1} \quad a_i \quad a_{i+1} \end{array} \right\} &= \sum_{b'} (F_{a_{i+1}}^{a_{i-1}22})_{b'}^{a_i} \delta_b^{b'} \left\{ \begin{array}{c} 2 \quad 2 \\ \diagdown \quad \diagup \\ b' \\ \hline a_{i-1} \quad a_{i+1} \end{array} \right\} \\ &= \sum_{a'_i} \left[(F_{a_{i+1}}^{a_{i-1}22})_b^{a'_i} \right]^* (F_{a_{i+1}}^{a_{i-1}22})_b^{a_i} \left\{ \begin{array}{c} 2 \quad 2 \\ | \quad | \\ \hline a_{i-1} \quad a'_i \quad a_{i+1} \end{array} \right\}, \end{aligned}$$

²This has a natural generalisation to other Hopf algebras. Condition (1) is unchanged while Condition (2) requires that the out-going anyon only remains of the same type. This modification is necessary when multiple copies of the same anyon appear after fusion.

provided the F -moves are unitary. Alternatively we can write this as [11, 24]

$$\tilde{P}_{i-1,i,i+1}^{(b)} = \sum_{a_{i-1}, a_i, a'_i, a_{i+1}} \left[\left(F_{a_{i+1}}^{a_{i-1}22} \right)_b^{a'_i} \right]^* \left(F_{a_{i+1}}^{a_{i-1}22} \right)_b^{a_i} |..a_{i-1}a'_ia_{i+1}.. \rangle \langle ..a_{i-1}a_ia_{i+1}..|.$$

As we expected this 2-site operator acts upon 3 labels in the anyon basis and leaves the first and last anyon invariant under its action.

The original isotropic or XXX Heisenberg local Hamiltonian was defined as the exchange interaction on neighbouring sites. This was generalised to allow the strength of the interaction for spins in the z -direction to differ to those in the x, y -direction resulting in the XXZ Hamiltonian below,

$$\begin{aligned} h &= \frac{J}{2} (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) + \frac{J_z}{2} (\sigma^z \otimes \sigma^z) + \left(\frac{J_z}{2} - J \right) I \otimes I, \\ &= \begin{pmatrix} J_z - J & 0 & 0 & 0 \\ 0 & -J & J & 0 \\ 0 & J & -J & 0 \\ 0 & 0 & 0 & J_z - J \end{pmatrix} \\ &= -2JP^{(-)} + (J_z - J)P^{(2)}, \end{aligned} \tag{3}$$

where σ^j are the usual Pauli matrices. This local Hamiltonian commutes with the action of D_3 as it is expressible in terms of projection operators. Furthermore we can use the natural anyon analogues of the projection operators to determine its equivalent operator in the anyon basis,

$$\tilde{h} = -2J\tilde{P}^{(-)} + (J_z - J)\tilde{P}^{(2)}.$$

This is equivalent to the local Hamiltonian for the ‘spin-1’ $su(2)_4$ model, up to a gauge transformation, mapping the anyons $(+, 2, -)$ to $(0, 1, 2)$. The anyons 0, 1 and 2 of $su(2)_4$ are the analogues of the spin-0, -1 and -2 particles of $su(2)$ [11, 24].

3 Quantum chains

To illustrate how the boundary conditions of a quantum chain affects the global symmetry we provide an account of a variety of models. For the spin chains we use models constructible via the QISM and its variants as these are commonly associated with quasi-triangular Hopf algebras. Open spin chains with free ends are seen to be in correspondence with open anyon chains while closed models of either type are more complicated as the global symmetry can be broken. We find that among the closed models *braided* models have a clear correspondence between the spin and anyonic formulations. It is then shown that while the periodic XXZ spin chain has an anyon counterpart, generic periodic spin chains do not. Likewise we show that generic periodic anyon chains have no spin chain counterparts, this includes D_3 anyons.

Open Chains

The simplest (and somewhat trivial) example of a direct equivalence between chains is the open chain with free ends (non-interacting boundary fields) case. Whatever symmetry is contained by the local Hamiltonian is inherited by the global Hamiltonian (using the condition of

coassociativity). The spin and anyon versions are of a very similar form,

$$\mathcal{H} = \sum_{i=1}^{\mathcal{L}-1} h_{i(i+1)} \quad \Leftrightarrow \quad \tilde{\mathcal{H}} = \sum_{i=1}^{\mathcal{L}-1} \tilde{h}_{(i-1)i(i+1)}. \quad (4)$$

These provide models with identical energy spectrums. The degeneracies of the each energy also match up once the spin dimension of each vector in the anyon basis is considered. Here we can see that the open XXZ chain is equivalent to the open D_3 chain or the ‘spin-1’ $su(2)_4$ chain restricted to the integer sector.

The introduction of non-trivial boundary fields will break the D_3 in either basis removing the correspondence between the two bases.

Braided Chains

Closed boundary models are more complicated due to the interaction between the first and last sites. One type of closed model which can be realised equivalently in both the spin and anyon bases are braided models [12, 14, 16]. These are guaranteed to have the full symmetry of the underlying algebra. In the case of the D_3 anyon chain, a braided model requires the existence of an invertible operator $b \in V_2 \otimes V_2$ satisfying³:

1. It is invertible and expressible in terms of projection operators of $\pi_2 \otimes \pi_2$, i.e. commutes with the action of the algebra on the 2-fold tensor product space,
2. It satisfies the braid equation, $b_{12}b_{23}b_{12} = b_{23}b_{12}b_{23}$,
3. It braids the local Hamiltonian, $h_{12}b_{23}b_{12} = b_{23}b_{12}h_{23}$.

Once such an operator is found we can define a global braiding operator and global Hamiltonian,

$$\mathcal{B} = b_{12}b_{23}\dots b_{(\mathcal{L}-1)\mathcal{L}} \quad \text{and} \quad \mathcal{H} = \mathcal{B}h_{(\mathcal{L}-1)\mathcal{L}}\mathcal{B}^{-1} + \sum_{i=1}^{\mathcal{L}-1} h_{i(i+1)}.$$

It follows that the global braiding operator and global Hamiltonian must commute with the action of the algebra. The global braiding operator plays the role of a generalised translation operator, satisfying,

$$\mathcal{B}h_{i(i+1)}\mathcal{B}^{-1} = h_{(i+1)(i+2)} \quad \text{and} \quad [\mathcal{B}, \mathcal{H}] = 0,$$

for $1 \leq i \leq \mathcal{L} - 2$. The additional term in this model, although it acts globally, is viewed as a local interaction as it commutes with all local Hamiltonians not acting on either site 1 or \mathcal{L} . Thus compared to the open chain the additional term only gives a finite correction to the energy. This model has a natural anyonic counterpart.

For the XXZ chain we find many different operators satisfying conditions 2 and 3, however, only one also satisfies condition 1. This operator corresponds to the representation of the universal R -matrix of D_3 and gives rise to the periodic spin chain, which we discuss in the next section. The other operators, satisfying conditions 2 and 3 but not condition 1, may correspond to different anyonic theories.

³These conditions have been adapted from [12, 14, 16] to construct a model with D_3 symmetry, that is also invariant under the action of the global braiding operator, \mathcal{B} , but not necessarily integrable.

The Periodic XXZ Spin Chain

The periodic XXZ chain can be realised in the anyon basis as it is also a braided model. This occurs because the permutation operator is also expressible in terms of projection operators and is consequently a suitable braiding operator, explicitly this is,

$$P = P^{(+)} - P^{(-)} + P^{(2)} \quad \text{where} \quad P(v \otimes w) = w \otimes v.$$

This allows the use of the braided model formalism to consider periodic XXZ spin chain in the anyon basis.

We remark that it is in general not possible to represent periodic spin chains in the anyon basis as periodicity can break the underlying symmetry. The breaking of this underlying symmetry is related to the (lack of) cocommutativity of the quasi-triangular Hopf algebra in question. However, irrespective of whether the symmetry is broken certain bulk properties including energy per site and the central charge are consistent with the open chain [2, 6].

The Periodic D_3 Anyon Chain

Now we consider the periodic D_3 anyon chain starting from the view point of an open chain. Using the \mathcal{L} sites with the additional auxiliary space, we have the global Hamiltonian given by Equation (4). We then impose periodicity in a basis sense, although the model itself will not be translationally invariant, by requiring that the in-coming anyon is equal to the out-going anyon, i.e. $a_0 = a_{\mathcal{L}}$. Thus we are only considering an invariant subspace of the full Hilbert space and now have that the auxiliary space is coupled to the rest. We can calculate both the anyon and spin dimensions

$$\begin{aligned} \text{Anyon dimension} &= \sum_a N_{\mathcal{L}}^{(aa)} = 2^{\mathcal{L}} + (-1)^{\mathcal{L}}, \\ \text{Spin dimension} &= \sum_a N_{\mathcal{L}}^{(aa)} \dim(V_a) = \frac{1}{3} [5 \cdot 2^{\mathcal{L}} + 2 \cdot (-1)^{\mathcal{L}}]. \end{aligned}$$

At this stage we have that $a_{\mathcal{L}}$ is still invariant under the action of the Hamiltonian and subsequently there still exist a corresponding model in the spin basis.

However, to obtain the periodic anyon models as presented in [10], which are translationally invariant, we need to include the term $\tilde{h}_{(\mathcal{L}-1)\mathcal{L}1}$. Once this term is included we no longer have that the out-going (now also in-coming) anyon is unchanged by the Hamiltonian implying that the D_3 symmetry in the spin sense is lost and this model has no spin model counterpart⁴⁵.

The periodic anyon boundary conditions for this model have yet to be studied in substantial detail. It follows that the ground-state energy per lattice site must be the same as the periodic spin case. Also, the central charge will match the periodic and open spin chains. It is of interest then to compare the low-lying excitations of the XXZ chain [3, 13] with their anyon counterparts.

⁴Alternatively we could have considered a $\mathcal{L} + 1$ site model and required $a_0 = a_{\mathcal{L}}$ and $a_1 = a_{\mathcal{L}+1}$. This would not have demonstrated as clearly how the D_3 invariance is lost.

⁵We remark that while from the perspective of this letter the D_3 symmetry has been lost there are other notions of D_3 symmetry which can be applied e.g. when the periodic anyon chain is viewed as living on a torus then eigenstates are classified by an associated flux, labeled by a D_3 anyon, through the torus, rather than by an out-going anyon [4, 10, 11].

4 Discussion

A correspondence between quantum spin and anyon chains exist when there is the underlying symmetry of a quasi-triangular Hopf algebra present. The symmetry inherited by the global Hamiltonian from the local Hamiltonian will depend upon the choice of boundary conditions. Open and braided models have a natural correspondence between the spin and anyon bases. On the other hand periodic models generally do not.

In the spin language the symmetry is present if the global Hamiltonian commutes with the action of the algebra. While in the anyon language we require the the in-coming and out-going anyons to be invariant under the action of the global Hamiltonian.

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References

- [1] *The On-Line Encyclopedia of Integer Sequences*, published electronically at <http://oeis.org>, (2011).
- [2] I. Affleck, *Universal Term in the Free Energy at a Critical Point and the Conformal Anomaly*, Phys. Rev. Lett., **56**, 746–748, (1986).
- [3] F.C. Alcaraz, M.N. Barber and M.T. Batchelor, *Conformal Invariance, the XXZ Chain and the Operator Content of Two- Dimensional Critical Systems*, Ann. Phys., **182**, 280–343, (1988).
- [4] E. Ardonne, J. Gukelberger, A.W.W. Ludwig, S. Trebst and M. Troyer, *Microscopic models of interacting Yang–Lee anyons*, New J. Phys., **13**, 045006, (2011).
- [5] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London, (1982).
- [6] H.W.J. Blöte, J.L. Cardy and M.P. Nightingale, *Conformal Invariance, the Central Charge, and Universal Finite-Size Amplitudes at Criticality*, Phys. Rev. Lett., **56**, 742–745, (1986).
- [7] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, (1994).
- [8] J.D. Cloizeaux and M. Gaudin, *Anisotropic Linear Magnetic Chain*, J. Math. Phys., **7**, 1384, (1966).
- [9] M. de Wild Propitius and F.A. Bais, *Discrete gauge theories*, In Particles and Fields, Eds. G. Semenoff and L. Vinet, CRM Series in Mathematical Physics (Springer-Verlag, New York), 353–353, (1998).
- [10] A. Feiguin, S. Trebst, A.W.W. Ludwig, M. Troyer, A.Y. Kitaev, Z. Wang and M.H. Freedman, *Interacting Anyons in Topological Quantum Liquids: The Golden Chain*, Physical Review Letters, **98**, 160409, (2007).

- [11] C. Gils, E. Ardonne, S. Trebst, D.A. Huse, A.W.W. Ludwig, M. Troyer, and Z. Wang, in preperation.
- [12] H. Grosse, S. Pallua, P. Prester and E. Raschhofer, *On a quantum group invariant spin chain with non-local boundary conditions*, J. Phys. A: Math. Gen., **27**, 4761-4771, (1994).
- [13] C.J. Hamer, *Finite-size corrections for ground states of the XXZ Heisenberg chain*, J. Phys. A: Math. Gen. **19**, 3335, (1986).
- [14] M. Karowski and A. Zapletal, *Quantum Group Invariant Integrable n-State Vertex Models with Periodic Boundary Conditions*, Nucl. Phys. B, **419**, 567-588, (1994).
- [15] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, (1993).
- [16] J. Links and A. Foerster, *On the construction of integrable closed chains with quantum supersymmetry*, J. Phys. A, **30**, 2483-2487, (1997).
- [17] S. Majid, *Foundations of quantum group theory*, Cambrigde University Press, (1995).
- [18] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys., **123**, 171-254, (1989).
- [19] J. Preskill, *Lecture Notes for Physics 219: Quantum Computation*, Lecture Notes California Institute of Technology, (2004).
- [20] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen., **21**, 2375-2389, (1988).
- [21] E.K. Sklyanin, L.A. Takhtadzhyan and L.D. Faddeev, *Quantum inverse problem method. 1*, Theor. Math. Phys., **40**, 688-706, (1979).
- [22] J.K. Slingerland and F.A. Bais, *Quantum groups and non-Abelian braiding in quantum Hall systems*, Nucl. Phys. B, **612**, 229-290, (2001).
- [23] S. Trebst, M. Troyer, Z. Wang and A.W.W. Ludwig, *A short introduction to Fibonacci anyon models*, Prog. Theor. Phys. Suppl., **176**, 384, (2008).
- [24] V. Verbus, L. Martina and A.P. Protogenov, *Chain of interacting $SU(2)_4$ anyons and quantum $SU(2)_k \times \overline{SU(2)_k}$ doubles*, Theoretical and Mathematical Physics, **167**, 843-855, (2011).
- [25] C.N. Yang and C.P. Yang, *One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe's Hypothesis for Ground State in a Finite System*, Phys. Rev., **150**, 321-327, (1966).

A F-moves and projection operators

We have calculated the F -moves though explicitly decomposing the space $V_a \otimes V_b \otimes V_c$ in the two different manners mentioned previously and by then looking at the transformations between them. The F -moves which deal with only one-dimensional irreps:

$$(F_{a \times b \times c}^{abc})_y^x = \delta_x^{a \times b} \delta_y^{b \times c}$$

where $a, b, c \in \{+, -\}$. The F -moves with precisely one 2-particle present

$$(F_2^{ab2})_y^x = \delta_x^{a \times b} \delta_y^2 \quad (F_2^{a2c})_y^x = \delta_x^2 \delta_y^2 \quad (F_2^{2bc})_y^x = \delta_x^2 \delta_y^{b \times c}$$

where $a, b, c \in \{+, -\}$. The F -moves with precisely two 2-particles present and one +-particle

$$\begin{aligned} (F_+^{+22})_y^x &= \delta_x^2 \delta_y^+ & (F_+^{2+2})_y^x &= \delta_x^2 \delta_y^2 & (F_+^{22+})_y^x &= \delta_x^+ \delta_y^2 \\ (F_-^{+22})_y^x &= \delta_x^2 \delta_y^- & (F_-^{2+2})_y^x &= \delta_x^2 \delta_y^2 & (F_-^{22+})_y^x &= \delta_x^- \delta_y^2 \\ (F_2^{+22})_y^x &= \delta_x^2 \delta_y^2 & (F_2^{2+2})_y^x &= \delta_x^2 \delta_y^2 & (F_2^{22+})_y^x &= \delta_x^2 \delta_y^2 \end{aligned}$$

Here are the F -moves with precisely two 2-particles present and one --particle

$$\begin{aligned} (F_+^{-22})_y^x &= \delta_x^2 \delta_y^- & (F_+^{2-2})_y^x &= -\delta_x^2 \delta_y^2 & (F_+^{22-})_y^x &= -\delta_x^- \delta_y^2 \\ (F_-^{-22})_y^x &= \delta_x^2 \delta_y^+ & (F_-^{2-2})_y^x &= -\delta_x^2 \delta_y^2 & (F_-^{22-})_y^x &= -\delta_x^+ \delta_y^2 \\ (F_2^{-22})_y^x &= -\delta_x^2 \delta_y^2 & (F_2^{2-2})_y^x &= \delta_x^2 \delta_y^2 & (F_2^{22-})_y^x &= -\delta_x^2 \delta_y^2 \end{aligned}$$

Here are the other F -moves with all 2-particles:

$$(F_+^{222})_y^x = \delta_x^2 \delta_y^2 \quad (F_-^{222})_y^x = -\delta_x^2 \delta_y^2$$

and

$$(F_2^{222})_y^x = \frac{1}{2}(\delta_x^+ \delta_y^+ - \delta_x^+ \delta_y^- + \delta_x^- \delta_y^+ - \delta_x^- \delta_y^-) + \frac{1}{\sqrt{2}}(\delta_x^+ \delta_y^2 - \delta_x^- \delta_y^2 + \delta_x^2 \delta_y^+ + \delta_x^2 \delta_y^-)$$

The projection operators are:

$$\begin{aligned} \tilde{P}_{(i-1)i(i+1)}^{(+)} &= n_{i-1}^+ n_{i+1}^+ + n_{i-1}^- n_{i+1}^- + \frac{1}{4} n_{i-1}^2 \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}_i n_{i+1}^2, \\ \tilde{P}_{(i-1)i(i+1)}^{(-)} &= n_{i-1}^+ n_{i+1}^- + n_{i-1}^- n_{i+1}^+ + \frac{1}{4} n_{i-1}^2 \begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 2 \end{pmatrix}_i n_{i+1}^2, \\ \tilde{P}_{(i-1)i(i+1)}^{(2)} &= n_{i-1}^+ n_{i+1}^2 + n_{i-1}^2 n_{i+1}^+ + n_{i-1}^- n_{i+1}^2 + n_{i-1}^2 n_{i+1}^- + \frac{1}{2} n_{i-1}^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_i n_{i+1}^2. \end{aligned}$$

We have adopted the notation that n_i^a projects on to anyon a for the i th label and the vector $(x, y, z)_i^T$ corresponds to $x |..+_i.. \rangle + y |..-_i.. \rangle + z |..2_i.. \rangle$.